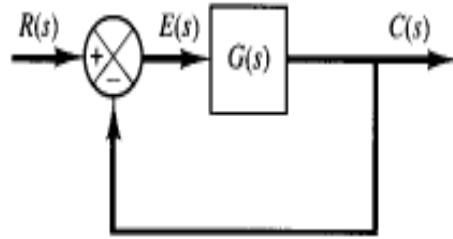


Steady State Error (e_{ss}):



It is the difference between actual input and desired output as time tends to infinity.

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{sR}{1 + GH}$$

Thus e_{ss} depends upon system input, open loop transfer function, and dominant nonlinearities present if any.

1. Static Position Error Constant k_p :

The e_{ss} of the system for a unit step input is

$$e_{ss} = \lim_{s \rightarrow 0} \frac{s}{1 + G} \frac{1}{s} = \frac{1}{1 + k_p}$$

where $k_p = \lim_{s \rightarrow 0} G(s)$

2. Static Velocity Error Constant k_v :

The e_{ss} of the system for a unit ramp input is

$$e_{ss} = \lim_{s \rightarrow 0} \frac{s}{1 + G} \frac{1}{s^2} = \lim_{s \rightarrow 0} \frac{1}{sG} = \frac{1}{k_v}$$

where $k_v = \lim_{s \rightarrow 0} sG(s)$

3. Static acceleration Error Constant k_a :

The e_{ss} of the system for a unit parabolic input is

$$e_{ss} = \lim_{s \rightarrow 0} \frac{s}{1 + G} \frac{1}{s^3} = \lim_{s \rightarrow 0} \frac{1}{s^2 G} = \frac{1}{k_a}$$

where $k_a = \lim_{s \rightarrow 0} s^2 G(s)$

Example:

The unity feedback system with open loop transfer function:

$$G(s) = \frac{k(s + 3.15)}{s(s + 1.5)(s + 0.5)}$$

Find: (a) type of the system (b) static error constants and (c) errors due to position, velocity, and acceleration inputs.

Solution: system with type one.

- For Step input: $k_p = \lim_{s \rightarrow 0} G(s) = \lim_{s \rightarrow 0} \frac{k(s+3.15)}{s(s+1.5)(s+0.5)} = \infty$

$$e_{ss} = \frac{R}{1+k_p} = 0$$

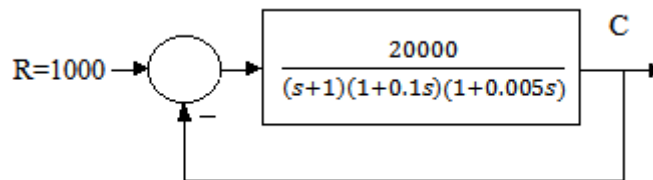
- For Ramp input: $k_v = \lim_{s \rightarrow 0} s G(s) = \lim_{s \rightarrow 0} \frac{s k(s+3.15)}{s(s+1.5)(s+0.5)} = 4.2 k$

$$e_{ss} = \frac{R}{k_v} = \frac{R}{4.2 k}$$

- For Parabolic input: $k_a = \lim_{s \rightarrow 0} s^2 G(s) = \lim_{s \rightarrow 0} s^2 \frac{k(s+3.15)}{s(s+1.5)(s+0.5)} = 0$

$$e_{ss} = \frac{R}{k_a} = \infty$$

Example: The heat treating oven is shown in figure below. The desired temperature is $R=1000^\circ\text{C}$. What is the steady state temperature?



Solution: $G(s) = \frac{20000}{(s+1)(1+0.1s)(1+0.005s)}$ $H = 1$ $R = \frac{1000}{s}$

$$k_p = \lim_{s \rightarrow 0} GH = \lim_{s \rightarrow 0} \frac{20000}{(s+1)(1+0.1s)(1+0.005s)} = 20000$$

$$e_{ss} = \frac{1000}{1 + k_p} = \frac{1000}{1 + 20000} = 0.04999$$

Steady state temperature $C_{ss} = \text{desired} - e_{ss} = 1000 - 0.04999 = 999.95^\circ\text{C}$

Example: For a system with open loop transfer function $GH = \frac{K}{s^2(s+2)(s+3)}$

Find the value of K to limit e_{ss} to 10 when system input is $r(t) = 1 + 10t + \frac{40}{2}t^2$

Solution: The input is combination of three inputs

$$e_{ss} = e_{ss1} + e_{ss2} + e_{ss3} = \frac{A_1}{1 + k_p} + \frac{A_2}{k_v} + \frac{A_3}{k_a}$$

$$k_p = \lim_{s \rightarrow 0} GH = \lim_{s \rightarrow 0} \frac{K}{s^2(s+2)(s+3)} = \infty$$

$$k_v = \lim_{s \rightarrow 0} sGH = \lim_{s \rightarrow 0} \frac{sK}{s^2(s+2)(s+3)} = \infty$$

$$k_a = \lim_{s \rightarrow 0} s^2GH = \lim_{s \rightarrow 0} \frac{s^2K}{s^2(s+2)(s+3)} = \frac{k}{6}$$

$$e_{ss} = \frac{1}{1 + \infty} + \frac{10}{\infty} + \frac{40}{k/6} \quad \therefore k = 24$$

Example: for a system with $GH = \frac{k(s+4)}{s(s^3+5s^2+6s)}$ Find static error due to input $\frac{A}{2}t^2$

Solution: type 2 system

$$k_a = \lim_{s \rightarrow 0} s^2GH = \lim_{s \rightarrow 0} \frac{s^2K(s+4)}{s(s^3+5s^2+6s)} = \frac{2K}{3}$$

$$e_{ss} = \frac{A}{k_a} = \frac{3A}{2K}$$

Dynamic Error Coefficients:

The static error suffers from the limitation that it allows determination of error coefficients for only three kinds of input. Also, it does not show how the error varies with time. These disadvantages can be overcome by finding generalized error coefficients that are often called **dynamic error coefficients**.

$$\frac{E}{R} = \frac{1}{1 + GH} = \frac{1}{k_1} + \frac{1}{k_2} s + \frac{1}{k_3} s^2 + \dots$$

Then

$$E = \frac{1}{k_1} R + \frac{1}{k_2} sR + \frac{1}{k_3} s^2 R + \dots$$

Taking $\mathcal{L}^{-1}$ then the **error series** is

$$e(t) = \frac{1}{k_1} r + \frac{1}{k_2} \dot{r} + \frac{1}{k_3} \ddot{r} + \dots$$

Where k_1, k_2 and k_3 are the **dynamic error coefficients**.

Example: The open loop transfer function of a servo system with unity feedback is given by $G = \frac{500}{s(1+0.1s)}$

(a) evaluate the error series of the system

(b) determine the e_{ss} for the input $r(t) = 1 + 2t + t^2$

Solution:

$$\frac{E}{R} = \frac{1}{1 + G} = \frac{s + 0.1s^2}{500 + s + 0.1s^2}$$

The long division gives $\frac{E}{R} = 0.002s + 0.000196s^2 - 0.000000792s^3 + \dots$

$$E = 0.002sR + 0.000196s^2R - 0.000000792s^3R + \dots$$

$$E = \frac{1}{k_1} R + \frac{1}{k_2} sR + \frac{1}{k_3} s^2 R + \frac{1}{k_4} s^3 R + \dots$$

Taking $\mathcal{L}^{-1}$

$$e(t) = 0.002\dot{r} + 0.000196\ddot{r} - 0.000000792\ddot{r} + \dots$$

With $r(t) = 1 + 2t + t^2$ then $\dot{r} = 2 + 2t$, $\ddot{r} = 2$ and $\ddot{r} = 0$

These gives

$$\begin{aligned} e(t) &= 0.002 (2 + 2t) + 0.000196 (2) - 0.000000792(0) \\ &= 0.004392 + 0.004t \end{aligned}$$

Therefore

$$k_1 = \frac{1}{0} = \infty$$

$$k_2 = \frac{1}{0.002} = 500$$

$$k_3 = \frac{1}{0.000196} = 5102.04$$

$$k_4 = \frac{1}{0.000000792} = -1262626.26$$

The $e_{ss} = \lim_{t \rightarrow \infty} (0.004392 + 0.004t) = \infty$

High Order Control Systems:

A high order system often contains less important poles that have little effect on system response. From given high order transfer function $M_H(s)$ we can find a low order transfer function $M_L(s)$ as an approximation that the two systems are similar according to some prescribed criterion.

Example:

$$M_H = \frac{C}{R} = \frac{8}{s^3 + 6s^2 + 12s + 8} = \frac{1}{1 + 1.5s + 0.75s^2 + 0.125s^3}$$

The closed loop poles are at $s=-2$, -2 and -2 .

Let $M_L = \frac{1}{1+d_1s+d_2s^2}$

Then $\frac{M_H}{M_L} = \frac{1+d_1s+d_2s^2}{1+1.5s+0.75s^2+0.125s^3}$

$$M_L(+s) = 1 + d_1s + d_2s^2$$

$$M_L(-s) = 1 - d_1s + d_2s^2$$

$$\begin{aligned} M_L(+s)M_L(-s) &= (1 + d_1s + d_2s^2)(1 - d_1s + d_2s^2) \\ &= 1 + (2d_2 - d_1^2)s^2 + d_2^2s^4 \end{aligned}$$

$$M_H(+s) = (1 + 1.5s + 0.75s^2 + 0.125s^3)$$

$$M_H(-s) = (1 - 1.5s + 0.75s^2 - 0.125s^3)$$

$$M_H(+s)M_H(-s) = 1 - 0.75s^2 - 0.1875s^4 - 0.015625s^6$$

For $\frac{M_H(+s)M_H(-s)}{M_L(+s)M_L(-s)} = 1$ then

$$(2d_2 - d_1^2) = -0.75 \quad \text{and} \quad d_2^2 = -0.1875$$

$$\therefore d_1 = 1.271 \quad \text{and} \quad d_2 = 0.433$$

$$M_L = \frac{1}{1 + 1.271s + 0.433s^2} = \frac{2.31}{s^2 + 2.94s + 2.31}$$

M_L has poles at $s_1 = -1.468 + j0.384$ and $s_2 = -1.468 - j0.384$

